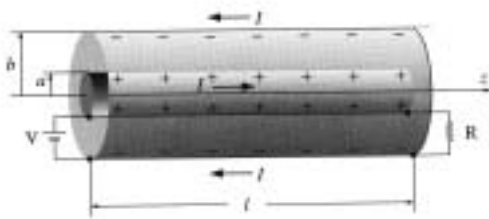


記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

1. (8%, 8%, 4%)

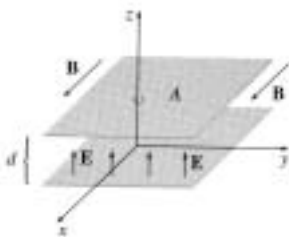
A long coaxial cable of length  $l$  consists of an inner conductor (radius  $a$ ) and an outer conductor (radius  $b$ ). It is connected to a battery at one end and a resistor at the other, as shown in the figure below. The inner conductor carries a uniform charge per unit length  $\lambda$ , and a steady current  $I$  to the right; the outer conductor has the opposite charge and current. [Hint: assume the two conductors are held at a potential difference  $V$ .]

- Calculate the  $\mathbf{E}$  and  $\mathbf{B}$  fields using Gauss's law and Ampere's law.
- Calculate the power (energy per unit time) transported down the cable.
- Calculate the energy density  $u_{\text{em}}$ .



2. (10%, 10%) A charged parallel-plates capacitor (with uniform electric field  $\mathbf{E} = E_0 \hat{\mathbf{z}}$ ) is placed in a uniform magnetic field  $\mathbf{B} = B_0 \hat{\mathbf{x}}$ , as shown in the figure.

- Show the equation of motion for a particle of charge  $q$  and mass  $m$ . [Hint: use the velocities as variables and write down the equations of motion in Cartesian coordinate].
- Solve the coupled differential equations. If the particle is initially at rest, describe how it moves. [Hint: This is the so-called  $\mathbf{E} \times \mathbf{B}$  drift]

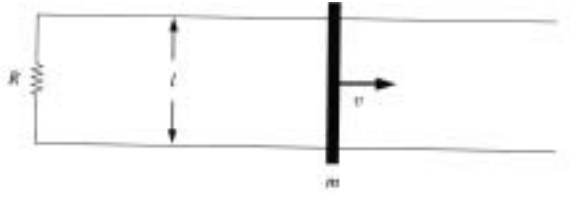


3. (6%, 7%, 7%) A metal bar of mass  $m$  slides frictionlessly on two parallel conducting rails a distance  $l$  apart (see figure below). A resistor  $R$  is connected across the rails and a uniform magnetic field  $\mathbf{B}$ , pointing into the page, fills the entire region.

- If the bar moves to the right at speed  $v$ , what is the current in the resistor? In what direction does it flow?
- What is the magnetic force on the bar? [Hint: to speed up or to slow down the movement].



(c) If the bar starts out with speed  $v_0$  at time  $t=0$ , and is left to slide, what is its speed at a later time  $t$ ?



4. (7%, 7%, 6%)

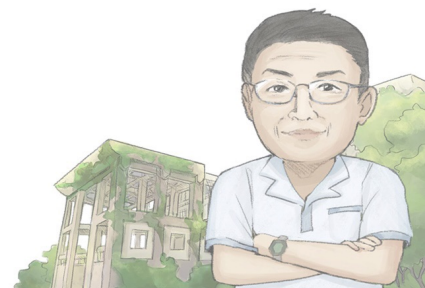
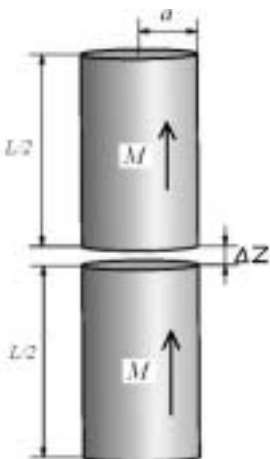
- Write down Maxwell's equations in terms of free charges  $\rho_f$  and current  $\mathbf{J}_f$ . Also for linear media, write the appropriate constitutive relations, giving  $\mathbf{D}$  and  $\mathbf{H}$  in terms of  $\mathbf{E}$  and  $\mathbf{B}$ .
- Write down the four boundary conditions ( $E^\perp$ ,  $E^\parallel$ ,  $B^\perp$ , and  $B^\parallel$ ) for linear media, if there is no free charge and no free current at the interface.
- Write down the equations for conservation of charge, energy, and momentum. Please explain the symbols you use as clear as possible.

5. (6%, 7%, 7%) A cylindrical magnet of length  $L$  and radius  $a$  carries a uniform magnetization  $\mathbf{M}$  parallel to its axis. It is then cut into two pieces of equal length.

- Determine the magnetic field  $\mathbf{B}$  in the gap.
- Find the force between these two magnets using the Maxwell stress tensor.
- Find the force between them using the concept that  $\mathbf{F} = -\nabla U$ .

[Hint 1: Maxwell's stress tensor  $T_{ij} \equiv \epsilon_0(E_i E_j - \frac{1}{2} \delta_{ij} E^2) + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2} \delta_{ij} B^2)$ ;

Hint 2: attractive or repulsive force?]



1. (a)

$$\text{Gauss's law: } \int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0}, \quad E 2\pi r l = \frac{\lambda l}{\epsilon_0} \Rightarrow \mathbf{E} = \frac{\lambda}{2\pi r \epsilon_0} \hat{\mathbf{r}}$$

$$\text{Ampere's law: } \oint \mathbf{B} \cdot d\ell = \mu_0 I, \quad B 2\pi r = \mu_0 I \Rightarrow \mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\boldsymbol{\phi}}$$

$$V = \int \mathbf{E} \cdot d\ell = \int_a^b \frac{\lambda}{2\pi r \epsilon_0} dr = \frac{\lambda}{2\pi \epsilon_0} \ln(b/a), \Rightarrow \lambda = \frac{2\pi \epsilon_0 V}{\ln(b/a)} \therefore \mathbf{E} = \frac{V}{r \ln(b/a)} \hat{\mathbf{r}}$$

(b)

$$\mathbf{S} \equiv \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0} \frac{V}{r \ln(b/a)} \frac{\mu_0 I}{2\pi r} \hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \frac{VI}{2\pi r^2 \ln(b/a)} \hat{\mathbf{z}}$$

$$P = \int \mathbf{S} \cdot d\mathbf{a} = \int_a^b \mathbf{S} \cdot d\mathbf{a} = \int_a^b \frac{VI}{2\pi r^2 \ln(b/a)} 2\pi r dr = VI$$

(c)

$$u_{\text{em}} = \frac{1}{2} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) = \frac{\epsilon_0}{2} \left( \frac{V}{r \ln(b/a)} \right)^2 + \frac{1}{2\mu_0} \left( \frac{\mu_0 I}{2\pi r} \right)^2$$

2. (a)

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \Rightarrow \begin{cases} m\dot{v}_x = 0 \\ m\dot{v}_y = qv_z B_0 \\ m\dot{v}_z = qE_0 - qv_y B_0 \end{cases}$$

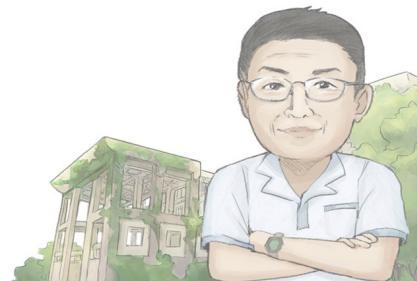
(b)

$x$  component:  $m\dot{v}_x = 0, v_x = \text{const.}$

$$y \text{ and } z \text{ components are coupled: } \begin{cases} m\dot{v}_y = qv_z B_0 \\ m\dot{v}_z = qE_0 - qv_y B_0 \end{cases}$$

$$\begin{cases} m\ddot{v}_y = q\dot{v}_z B_0 = \frac{qB_0}{m} (qE_0 - qv_y B_0) \\ m\ddot{v}_z = -q\dot{v}_y B_0 = -\frac{qB_0}{m} qv_z B_0 \end{cases} \Rightarrow \begin{cases} \ddot{v}_y = \left( \frac{qB_0}{m} \right)^2 \left( \frac{E_0}{B_0} - v_y \right) \\ \ddot{v}_z = -\left( \frac{qB_0}{m} \right)^2 v_z \end{cases}$$

$$\text{Let } \left( \frac{qB_0}{m} \right) \equiv \omega_c \quad \begin{cases} \ddot{v}_y = -\omega_c^2 \left( v_y - \frac{E_0}{B_0} \right) \\ \ddot{v}_z = -\omega_c^2 v_z \end{cases} \Rightarrow \begin{cases} v_y = A \cos(\omega_c t + \phi_1) + \frac{E_0}{B_0} \\ v_z = B \cos(\omega_c t + \phi_2) \end{cases}$$



$$\text{Initial condition } (t=0): \begin{cases} (v_x, v_y, v_z) = (0, 0, 0) \\ (\dot{v}_x, \dot{v}_y, \dot{v}_z) = (0, 0, \frac{qE_0}{m}) \end{cases} \begin{cases} v_y = A \cos(\omega_c t + \phi_1) + \frac{E_0}{B_0} \\ v_z = B \cos(\omega_c t + \phi_2) \end{cases}$$

x component:  $v_x = 0$ .

$$y \text{ component: } \begin{cases} v_y = A \cos(\phi_1) + \frac{E_0}{B_0} = 0 \\ \dot{v}_y = -A\omega_c \sin(\phi_1) = 0 \end{cases} \Rightarrow v_y = -\frac{E_0}{B_0} \cos(\omega_c t) + \frac{E_0}{B_0}$$

$$z \text{ component: } \begin{cases} v_z = B \cos(\phi_2) = 0 \\ \dot{v}_z = -B\omega_c \sin(\phi_2) = \frac{qE_0}{m} \Rightarrow v_z = \frac{E_0}{B_0} \sin(\omega_c t) \end{cases}$$

The Larmor motion is the same as before, but there is superimposed a drift of the guiding center in the y direction. The particle moves in the y-z plane. This is the so-called **ExB** drift.

3.

$$(a) \text{ The emf: } \varepsilon = -\frac{d\phi}{dt} = -Bl \frac{dx}{dt} = -Blv = IR \Rightarrow I = -\frac{Blv}{R} \text{ (to compensate the change of flux)}$$

The current flow is counter-clockwise.

(b)

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = I\mathbf{l} \times \mathbf{B} = -\frac{B^2 l^2 v}{R} \leftarrow \text{(to the left, to slow down the flux change)}$$

(c)

$$F = m \frac{dv}{dt} = -\frac{B^2 l^2}{R} v \Rightarrow v = v_0 e^{-\frac{B^2 l^2}{mR} t}$$

4.

$$(a) \quad \begin{aligned} \nabla \cdot \mathbf{D} &= \rho_f & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J}_f \end{aligned}$$

$$\text{where } \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0(1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E} \text{ and } \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \Rightarrow \mathbf{B} = \mu_0(1 + \chi_m) \mathbf{H} = \mu \mathbf{H}$$

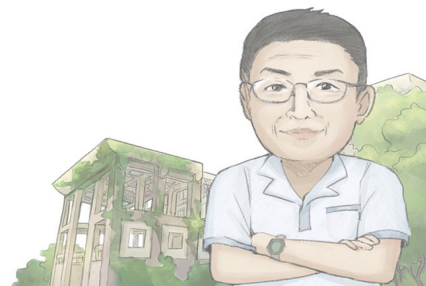
(b) For linear media,  $\mathbf{D} = \varepsilon \mathbf{E}$  and  $\mathbf{B} = \mu \mathbf{H}$

$$\begin{aligned} D_1^\perp - D_2^\perp = 0 & \quad \mathbf{E}_1'' - \mathbf{E}_2'' = 0 \\ B_1^\perp - B_2^\perp = 0 & \quad \mathbf{H}_1'' - \mathbf{H}_2'' = 0 \end{aligned} \Rightarrow \begin{aligned} \varepsilon_1 E_1^\perp - \varepsilon_2 E_2^\perp &= 0 & \mathbf{E}_1'' - \mathbf{E}_2'' &= 0 \\ B_1^\perp - B_2^\perp &= 0 & \frac{1}{\mu_1} \mathbf{B}_1'' - \frac{1}{\mu_2} \mathbf{B}_2'' &= 0 \end{aligned}$$

(c) Conservation of charge

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}, \text{ where } \rho \text{ is the charge density and } \mathbf{J} \text{ is the current density.}$$

Conservation of energy



$$\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \mathbf{S},$$

where  $u_{\text{mech}}$  is the mechanical energy density,  $u_{\text{em}}$  is the electromagnetic energy density, and  $\mathbf{J}$  is the Poynting vector.

Conservation of momentum

$$\frac{\partial}{\partial t}(\mathbf{g}_{\text{mech}} + \mathbf{g}_{\text{em}}) = -\nabla \cdot (-\vec{\mathbf{T}}),$$

where  $\mathbf{g}_{\text{mech}}$  is the mechanical momentum density,  $\mathbf{g}_{\text{em}}$  is the electromagnetic momentum flux density, and  $\vec{\mathbf{T}}$  is the Maxwell stress tensor.

5.

(a) The boundary condition:

$$B_{\text{in}}^{\perp} - B_{\text{out}}^{\perp} = 0, \quad \frac{1}{\mu_{\text{in}}} \mathbf{B}_{\text{in}}^{\parallel} - \frac{1}{\mu_{\text{out}}} \mathbf{B}_{\text{out}}^{\parallel} = 0, \quad \text{where } \mathbf{B}_{\text{in}}^{\parallel} = 0 \text{ and } B_{\text{in}}^{\perp} = \mu_0 M$$

So  $\mathbf{B}_{\text{out}}^{\parallel} = 0$  and  $B_{\text{out}}^{\perp} = \mu_0 M$  in  $z$  direction.

(b)

$$T_{ij} \equiv \epsilon_0 (E_i E_j - \frac{1}{2} \delta_{ij} E^2) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} B^2)$$

$$E_x = E_y = E_z = 0 \quad \text{and} \quad B_x = B_y = 0$$

$$B_z = \mu_0 M \text{ at the bottom surface}$$

$$B_z = 0 \text{ at the top surface}$$

$$T_{xx} = T_{yy} = -T_{zz} = -\frac{1}{2\mu_0} B_z^2, \quad \text{other } T_{ij} = 0$$

$$\mathbf{F} = \oint_S \vec{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S} d\tau = \oint_S \vec{\mathbf{T}} \cdot d\mathbf{a}, \quad \text{since } \mathbf{S} = 0.$$

From the symmetric consideration, only  $z$  component is nonvanished.

Consider the upper half:

$$F_y = \int_{\text{top}} T_{zz} d\mathbf{a} + \int_{\text{bottom}} T_{zz} d\mathbf{a} + \int_{\text{cylinder}} T_{zz} d\mathbf{a} = \int_{\text{bottom}} T_{zz} d\mathbf{a} = \frac{\mu_0 M^2}{2} \pi a^2 \quad (\text{attractive force})$$

(c)

Consider a very small gap.

$$U = U_0 + \frac{1}{2\mu_0} (\mu_0 M)^2 (\pi a^2 \cdot \Delta z) = U_0 + \frac{\pi a^2 \mu_0}{2} M^2 \Delta z$$

$$\mathbf{F} = -\nabla U = -\frac{\pi a^2 \mu_0}{2} M^2 \hat{\mathbf{z}}, \quad \text{same as (b). (attractive force)}$$

